Set-Membership Identification of Wiener models with noninvertible nonlinearity

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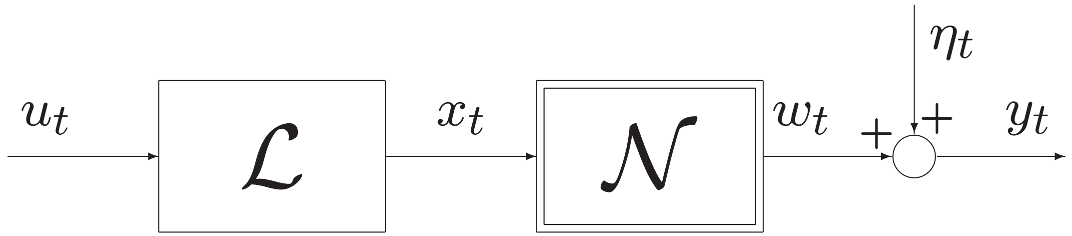
Wiener models

Wiener models consist of a linear dynamic system followed by a static nonlinearity



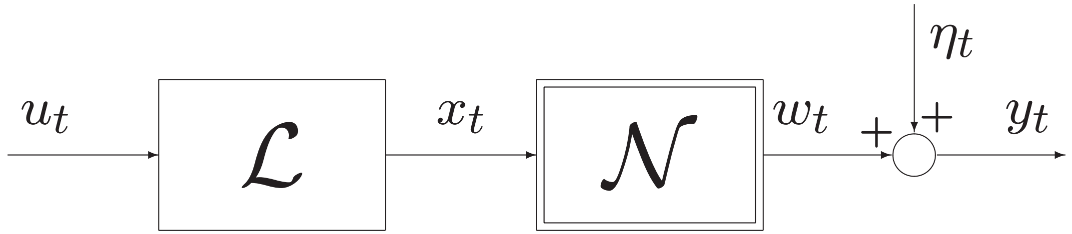
where:

$\mathcal{N}$: static (i.e. memoryless) nonlinearity
$\mathcal{L}$: linear subsystem
x_t: inner signal not measurable

Identification of Wiener models

- Motivations:
  - Main attractive feature: ability to embed process structure knowledge (e.g. nonlinearity in the measurement equipment).
  - Applications: nonlinear filtering, acoustic echo cancellation, identification of biological systems, modelling of electrical stimulated muscles, modeling of distillation columns ... many more

Problem formulation



$$w_t = \sum_{k=1}^{n} \gamma_k x_t^k$$

$$A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_{na} q^{-na}$$

$$B(q^{-1}) = b_0 + b_1 q^{-1} + \ldots + b_{nb} q^{-nb}$$

$$q^{-1} x_t = x_{t-1}$$

$$y_t = w_t + \eta_t$$

Problem formulation

- **Aim:** compute bounds on the parameters $\gamma^{\top} = [\gamma_1\, \gamma_2 \ldots \gamma_n]$, $\theta^{\top} = [a_1 \,..\, a_{na}\, b_0 \,..\, b_{nb}]$.

- **Prior assumption on the system:**

  1. stability;
  2. n, na and nb are known;
  3. the steady-state gain is not zero;
  4. a rough upper bound on the settling time of the system is known;

- **Prior assumption on the measurement uncertainty:**

  1. $\{\eta_t\}$ is UBB: $\|\{\eta_t\}\|_\infty \leq \Delta\eta_t$;
  2. $\Delta\eta_t$ is known;

Proposed solution: preliminary

Three-stage solution:

- First stage: computation of bounds on the nonlinear block parameters γ.
- Second stage: computation of bounds on the inner (unmeasurable) signal x_t.
- Third stage: computation of bounds on the linear block parameters.

Proposed solution: preliminary

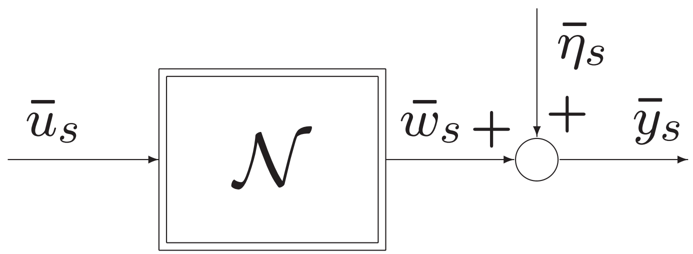
Remark 1: The parameterization is not unique

$\Rightarrow$ assume $g_{dc} = \dfrac{\sum_{j=0}^{nb} b_j}{1 + \sum_{i=1}^{na} a_i} = 1$

Remark 2: Stimulate the system with a set of step inputs with different amplitudes

$$\Downarrow$$

Steady-state operating conditions:



1st Stage: assessment of tight bounds on γ

- Get $M \geq n$ *steady-state* measurements:

$$\bar{y}_s = \sum_{k=1}^{n} \gamma_k \bar{u}_s^k + \bar{\eta}_s, \quad s = 1, \ldots, M$$

- The *feasible parameter set* of the nonlinear block, is defined as:

$$\mathcal{D}_\gamma = \{\gamma \in R^n : \bar{y}_s = \sum_{k=1}^{n} \gamma_k \bar{u}_s^k + \bar{\eta}_s, |\bar{\eta}_s| \leq \Delta\bar{\eta}_s; \; s = 1, \ldots, M\}$$

- $\mathcal{D}_\gamma$ is a convex polytope:
  (a) Provides tight bounds on parameters γ
  (b) Can be computed through standard algorithm from the SM literature

1st Stage: orthotope-outer bounding set $\mathcal{B}_\gamma$ containing $\mathcal{D}_\gamma$

The shape of $\mathcal{D}_\gamma$ may result quite complex for large M and n

$\Rightarrow$ Compute a tight orthotope outer-bound:

$$\mathcal{B}_\gamma = \{\gamma \in R^n : \gamma_j = \gamma_j^c + \delta\gamma_j, \\ \mid \delta\gamma_j \mid \leq \Delta\gamma_j/2, j = 1, \ldots, n\}$$

$$\gamma_j{}^c = \frac{\gamma_j^{min} + \gamma_j^{max}}{2}$$

$$\Delta\gamma_j = \mid \gamma_j^{max} - \gamma_j^{min} \mid$$

$$\gamma_j^{min} = \min_{\gamma \in \mathcal{D}_\gamma} \gamma_j \qquad \gamma_j^{max} = \max_{\gamma \in \mathcal{D}_\gamma} \gamma_j$$

Computational aspects: $\mathcal{B}_\gamma$ is obtained solving $2n$ LP problems with n variables and $2M$ constraints.

2nd Stage: bounds on the inner signal x_t

Simplified case: exactly known γ

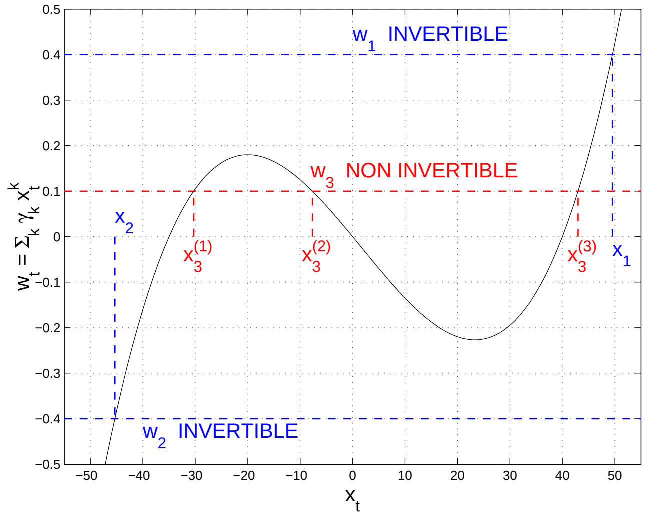
- Task: for given $\{w_t\}$ compute $\{x_t\}$, solving:

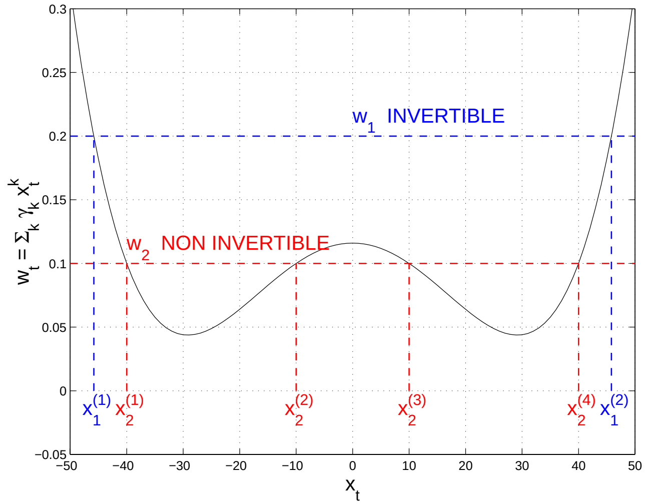
$$w_t - \sum_{k=1}^{n} \gamma_k x_t^k = p_t(x_t, w_t) = 0$$

- Problem: polynomials are noninvertible
- Key idea: design $\{u_t\}$ to drive $\{x_t\} \subset \Lambda^*$ such that:

$$\begin{cases} p_t(x_t, w_t) = 0 \\ x_t \in \Lambda^* \end{cases} \qquad (1)$$

has a unique solution for each t





2nd Stage: bounds on the inner signal x_t

Definition 1

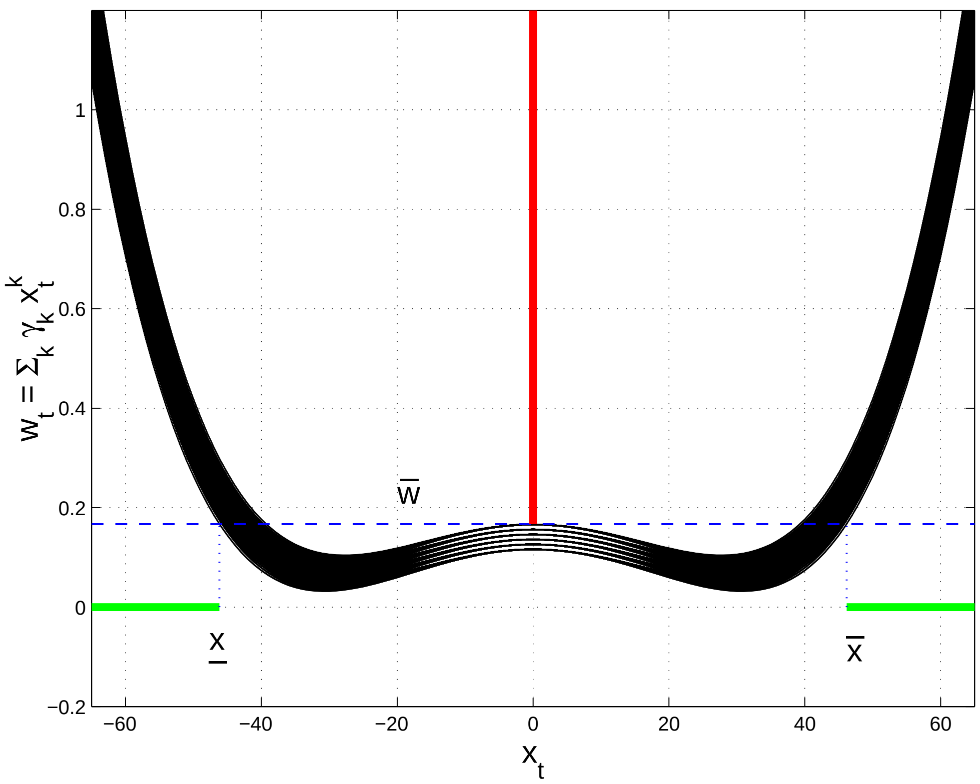
The set $W \subset R$ is an **Output Invertibility Interval** if for $w_t \in W$ each $p_t(x_t, w_t, \gamma) \in \Pi_t$ shows either only one real root (n odd) or two real roots when (n even).

Definition 2

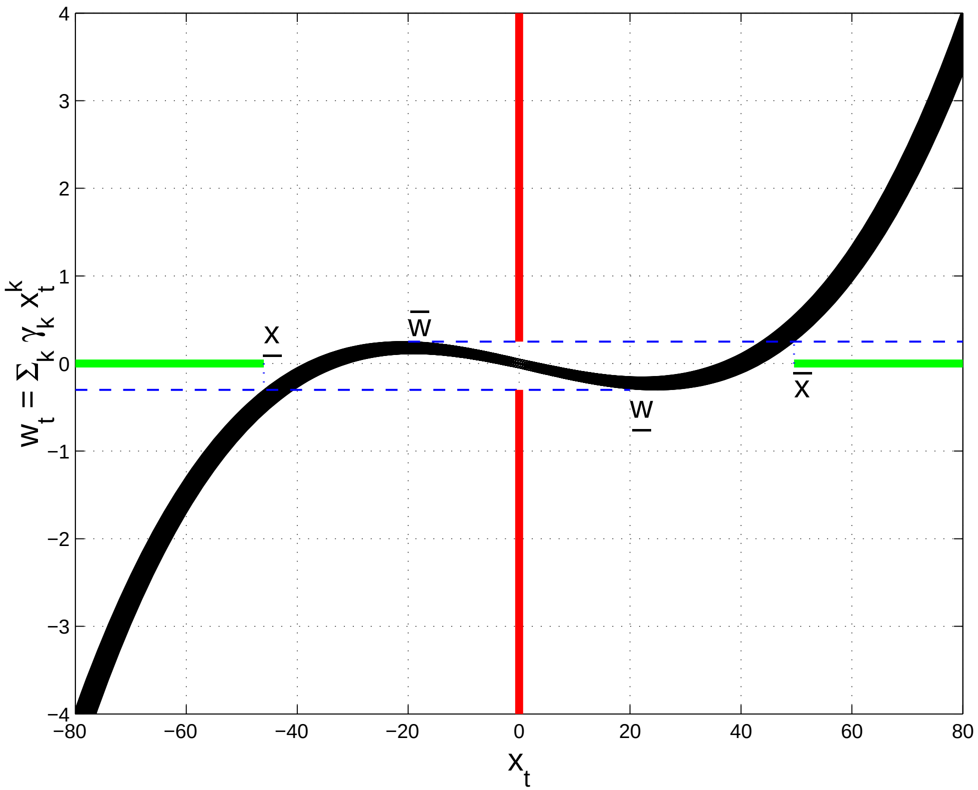
The set $X \subset R$ is a **Feasible Inner-signal Interval** if the set of output values $\mathcal{O} = \{w_t \in R : w_t = \mathcal{N}(x_t, \gamma),\ \mathcal{N}(x_t, \gamma) \in \mathcal{V}_t,\ x_t \in X\}$ is an *Output Invertibility Interval*

2nd Stage: bounds on the inner signal x_t

<u>*Case n even*</u>



<u>*Case n odd*</u>



2nd Stage: bounds on the inner signal x_t

Proposition 1 (Output Invertibility Intervals)

$\mathcal{N}(x_t, \gamma)$ with $\gamma \in \mathcal{D}_\gamma$, shows the following Output Invertibility Intervals (case n odd):

$$\overline{W} =]\overline{w}, +\infty[\text{ and } \underline{W} =]-\infty, \underline{w}[$$

where

$$\overline{w} = \max_{x_t \in \Upsilon_t} \max_{\gamma \in \mathcal{D}_\gamma} \sum_{k=1}^{n} \gamma_k x_t^k, \quad \underline{w} = \min_{x_t \in \Upsilon_t} \min_{\gamma \in \mathcal{D}_\gamma} \sum_{k=1}^{n} \gamma_k x_t^k$$

$$\Upsilon_t = \left\{ x_t \in R : \frac{d}{dx_t} \sum_{k=1}^{n} \gamma_k x_t^k = 0, \text{ for some } \gamma \in \mathcal{D}_\gamma \right\}$$

2nd Stage: bounds on the inner signal x_t

Proposition 2 (Feasible Inner-signal Intervals)

The Wiener system with uncertain $\mathcal{N}(x_t, \gamma)$, shows the following Feasible Inner-signal Intervals (case n odd and $\gamma_n > 0$):

$$\overline{X} =]\overline{x}, +\infty[\quad \text{and} \quad \underline{X} =]-\infty, \underline{x}[$$

where

$$\overline{x} = \max \left\{ x_t \in R : \overline{w} - \sum_{k=1}^{n} \gamma_k x_t^k = 0, \text{ for some } \gamma \in \mathcal{D}_\gamma \right\}$$

$$\underline{x} = \min \left\{ x_t \in R : \underline{w} - \sum_{k=1}^{n} \gamma_k x_t^k = 0, \text{ for some } \gamma \in \mathcal{D}_\gamma \right\}. \quad (2)$$

2nd Stage: Input Sequence Design

$u_t = u_{DC} + u_{td}$ such that $x_t = x_{DC} + x_{td} \in$ Feasible Inner-signal Interval
$g_{dc} = 1 \Rightarrow u_{DC} = x_{DC}$

Proposition 3 (Input sequence design)

- For a given $u_{DC} \geq \overline{x}$, each sample of the sequence $\{x_t\}$ belongs to $\overline{X}$ if:

$$\|\{u_{td}\}\|_\infty \leq \frac{|u_{DC} - \overline{x}|}{h_{up}}$$

- For given $u_{DC} \leq \underline{x}$, each sample of the sequence $\{x_t\}$ belongs to $\underline{X}$ if:

$$\|\{u_{td}\}\|_\infty \leq \frac{|u_{DC} - \underline{x}|}{h_{up}}$$

where: $\|h\|_1 \leq h_{up}$; h: impulse response the linear block; $\|h\|_1$: ℓ_1 norm of the linear block; $\|\cdot\|_\infty$: ℓ_∞ norm of a sequence.

2nd Stage: Input Sequence Design

Proposition 4 (Input signal design)
All the samples of $\{w_t\}$ belong to W^* if the samples of $\{y_t\}$ satisfy the following inequalities (case n odd):

$$y_t > \overline{y}\ \forall t \ \text{ or } \ y_t < \underline{y}\ \forall t, \quad \text{when } n \text{ is odd} \tag{3}$$

where

$$\overline{y} = \overline{w} + \Delta\eta_t, \qquad \underline{y} = \underline{w} - \Delta\eta_t$$

■

No bound h_{up} is *a priori* known

$$\Downarrow$$

Tune amplitude of $\{u_{td}\}$ until $\{y_t\}$ satisfies (3)

2nd Stage: bounds on $\{x_t\}$

Proposition 5 (Inner-signal bounds)

Given:

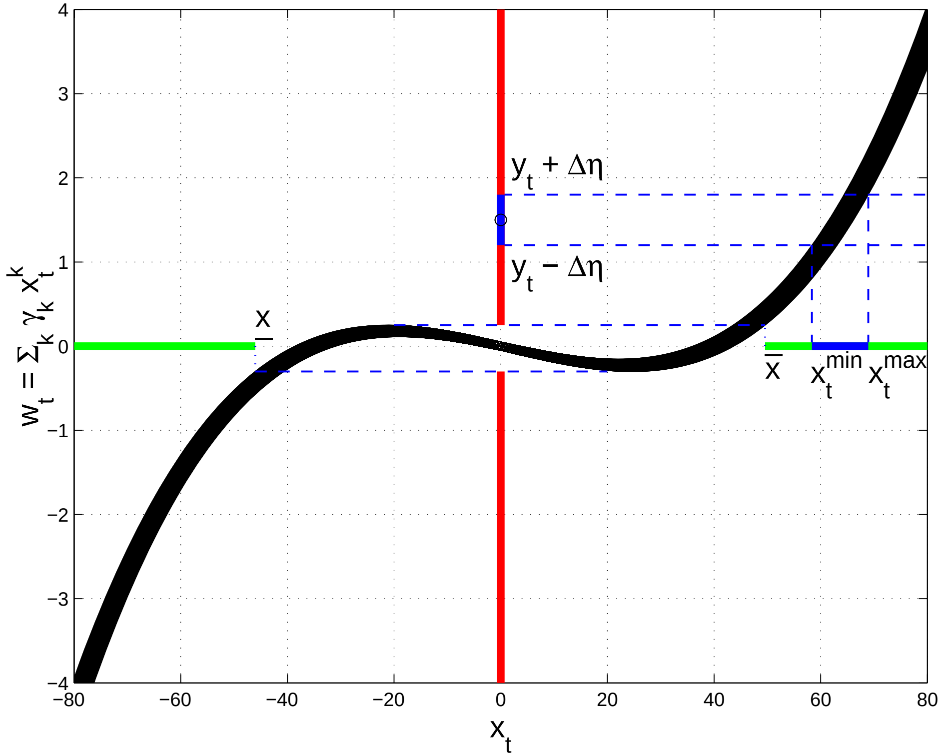
- $\mathcal{N}(x_t, \gamma)$ with $\gamma \in \mathcal{D}_\gamma$
- $\{u_t\}$ which drives $\{x_t\}$ into X^*
- the measured output sequence $\{y_t\}$

$$\Downarrow$$

Each sample x_t of $\{x_t\}$ is bounded as follows:

$$x_t^{min} \leq x_t \leq x_t^{max} \qquad (4)$$

where (case $\gamma_n > 0$ and $X^* = \overline{X}$):



$$x_t^{min} = \min \left\{ x_t \in R : y_t - \Delta\eta_t - \sum_{k=1}^{n} \gamma_k x_t^k = 0, \text{ for some } \gamma \in \mathcal{D}_\gamma \right\}$$

$$x_t^{max} = \max \left\{ x_t \in X^* : y_t + \Delta\eta_t - \sum_{k=1}^{n} \gamma_k x_t^k = 0, \text{ for some } \gamma \in \mathcal{D}_\gamma \right\}$$

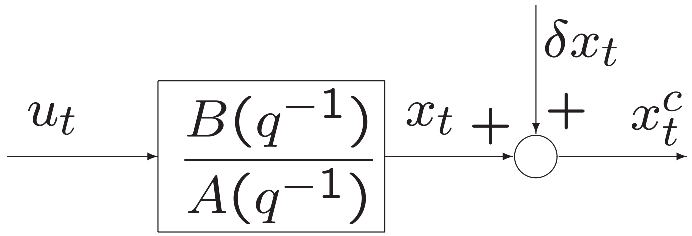
3rd stage: bounds on θ

Given the designed input sequence $\{u_t\}$, the uncertain inner sequence $\{x_t\}$ we define:

$$x_t^c = \frac{x_t^{min} + x_t^{max}}{2}$$

$$\mid \delta x_t \mid \leq \Delta x_t$$

$$\Delta x_t = \frac{x_t^{max} - x_t^{min}}{2}$$



$$\Downarrow$$

Output Error problem with UBB errors (special case of EIV problem)

(**V. Cerone - "Feasible parameter set of linear models with bounded errors in all variables"**, *Automatica '93*)

2nd Stage: Computational aspects

Computation of Υ_t:
find the roots of the uncertain polynomial:

$$p_t{}'(x_t,\gamma) = \frac{d}{dx_t}\sum_{k=1}^{n}\gamma_k x_t^k = -\sum_{k=1}^{n}k\gamma_k x_t^{k-1}$$

Proposed approach: gridding on $x_t \rightarrow$ one LP problem for each x_t.

Computation of $\overline{w}$ and $\underline{w}$:
solve the following two nonlinear programming problems:

$$\overline{w} = \max_{x_t\in\Upsilon_t}\max_{\gamma\in\mathcal{D}_\gamma}\sum_{k=1}^{n}\gamma_k x_t^k \quad \text{and} \quad \underline{w} = \min_{x_t\in\Upsilon_t}\min_{\gamma\in\mathcal{D}_\gamma}\sum_{k=1}^{n}\gamma_k x_t^k$$

Proposed approach: gridding on the set $\Upsilon_t \rightarrow$ one LP problem for each x_t.

2nd Stage: Computational aspects

Computation of $\overline{x}$ and $\underline{x}$:

solve the following problems (n odd, $\gamma_n > 0$):

$$\overline{x} = \max\{x_t \in R : p_t(x_t, \gamma, \overline{w}) = 0, \text{ for some } \gamma \in \mathcal{D}_\gamma\} \quad (5)$$

and

$$\underline{x} = \min\{x_t \in R : p_t(x_t, \gamma, \underline{w}) = 0, \text{ for some } \gamma \in \mathcal{D}_\gamma\} \quad (6)$$

Computation of x_t^{max} and x_t^{min}:

solve the following problems:

$$x_t^{max} = \max\{x_t \in \overline{X} : p_t(x_t, \gamma, y_t + \Delta\eta_t) = 0, \text{ for some } \gamma \in \mathcal{D}_\gamma\} \quad (7)$$

$$x_t^{min} = \min\{x_t \in R : p_t(x_t, \gamma, y_t - \Delta\eta_t) = 0, \text{ for some } \gamma \in \mathcal{D}_\gamma\} \quad (8)$$

2nd Stage: Computational aspects

Algorithm 1 (Computation of $\overline{x}$)

```
1.  Set α = α_0 and
ε = prescribed tolerance.
2.  Compute
r = max{x_t ∈ R : p_t^nom(x_t, γ^c, w̄) = 0}.
3.  Set x_m = r.
4.  Set x_M = x_m + α.
5.  If ∃γ^◇ ∈ D_γ : p_t(x_M, γ^◇, w̄) = 0 then
        x_m = x_M;
    else
        If |x_M − x_m| < ε then
                x̄_* = x_M;
                return x̄_*;
                stop algorithm.
        else
                α = α/2;
        end if
    end if.
8.  Repeat from 4.
```

Main properties:

1. Algorithm 1 is convergent.
2. Algorithm 1 provides an upper bound $\overline{x}_*$ of $\overline{x}$ with $|\overline{x}_* - \overline{x}| \leq \epsilon$.
3. Step 5 is a LP problem.

Remark: Algorithm 1 can be also used to compute $\underline{x}$, x_t^{max} and x_t^{min}

Example:

Parameters of the simulated system:

γ_1 =-5, γ_2 =-4, $\gamma_3 = 1$;
$A(q^{-1}) = (1 - 1.1q^{-1} + 0.28q^{-2})$;
$B(q^{-1}) = (0.1q^{-1} + 0.08q^{-2})$

Signal to noise ratio:

$$SNR = 10 \log \left\{ \sum_{t=1}^{N} w_t^2 \Big/ \sum_{t=1}^{N} \eta_t^2 \right\}$$

Measurement output errors:
Bounded absolute output errors have been considered:

$|\ \eta_t\ | \leq \Delta\eta_t$; η_t belongs to the uniform distribution $U[-\Delta\eta_t, +\Delta\eta_t]$
$|\ \bar{\eta}_s\ | \leq \Delta\bar{\eta}_s$; $\bar{\eta}_s$ belongs to the uniform distribution $U[-\Delta\bar{\eta}_s, +\Delta\bar{\eta}_s]$

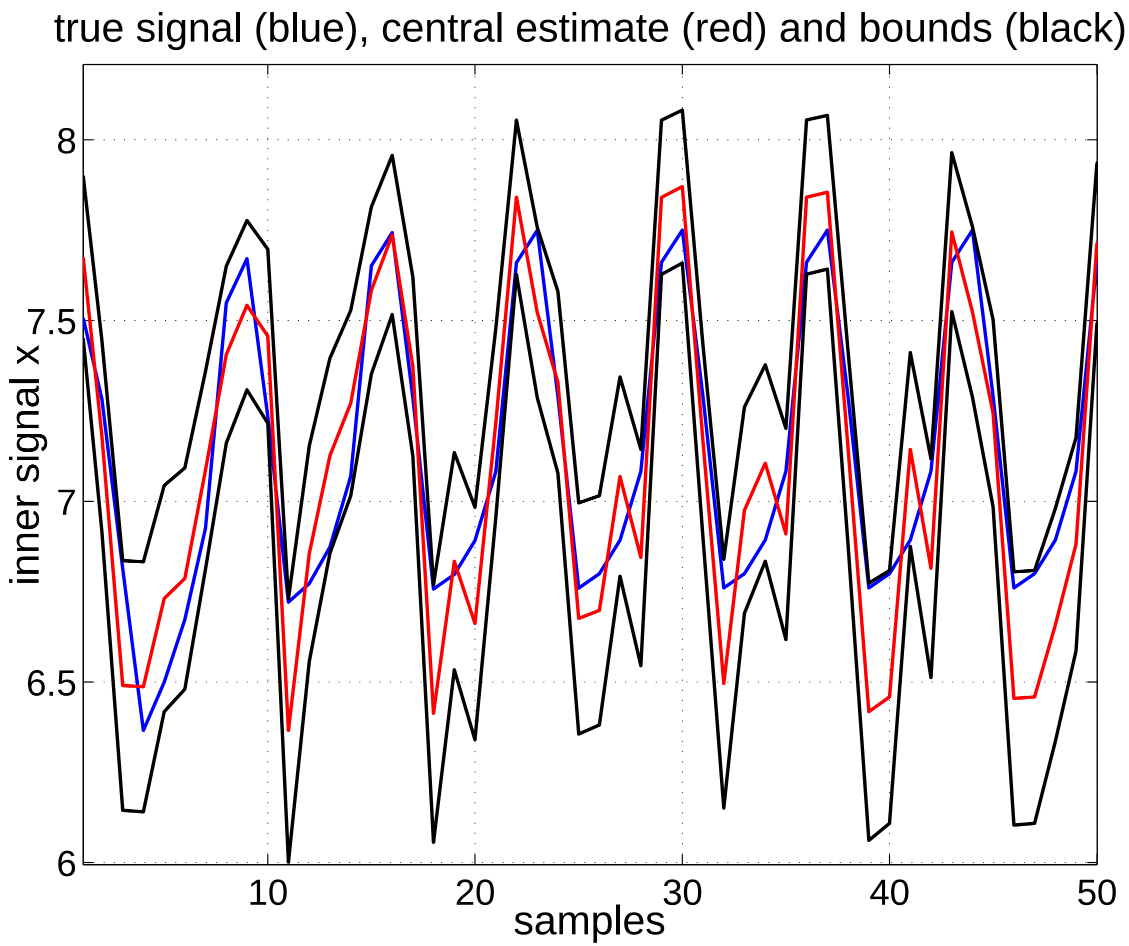
$\Delta\eta_t = \Delta\bar{\eta}_s \doteq \Delta\eta$

Nonlinear block parameter: central estimates and parameters uncertainty bounds

Number of steady-state samples: $M = 50$

$\overline{SNR}$ (dB)	γ_j	True Value	γ_j^c	$\Delta\gamma_j$
58.2	γ_1	-5.000	-4.999	2.1e-3
	γ_2	-4.000	-4.000	1.8e-4
	γ_3	1.000	1.000	4.8e-5
38.2	γ_1	-5.000	-5.027	3.6e-2
	γ_2	-4.000	-3.995	8.1e-3
	γ_3	1.000	1.001	1.6e-3
18.4	γ_1	-5.000	-5.101	1.1e-1
	γ_2	-4.000	-4.000	1.0e-2
	γ_3	1.000	1.004	5.1e-3

Inner signal estimate



Linear block parameters: central estimates and parameters uncertainty bounds

Number of transient samples: $N = 100$

SNR (dB)	θ_j	True Value	θ_j^c	$\Delta\theta_j$
58.2	θ_1	-1.100	-1.100	5.3e-3
	θ_2	0.280	0.280	5.1e-3
	θ_3	0.100	0.100	6.1e-4
	θ_4	0.080	0.080	5.6e-4
38.0	θ_1	-1.100	-1.106	7.9e-2
	θ_2	0.280	0.288	7.4e-2
	θ_3	0.100	0.100	8.0e-3
	θ_4	0.080	0.081	9.0e-3
18.2	θ_1	-1.100	-1.211	3.9e-1
	θ_2	0.280	0.403	3.6e-1
	θ_3	0.100	0.099	4.2e-2
	θ_4	0.080	0.101	4.7e-2

Number of transient samples: $N = 1000$

SNR (dB)	θ_j	True Value	θ_j^c	$\Delta\theta_j$
58.2	θ_1	-1.100	-1.100	1.9e-3
	θ_2	0.280	0.280	1.8e-3
	θ_3	0.100	0.100	1.9e-4
	θ_4	0.080	0.080	2.2e-4
38.4	θ_1	-1.100	-1.102	5.8e-2
	θ_2	0.280	0.282	5.4e-2
	θ_3	0.100	0.100	6.1e-3
	θ_4	0.080	0.079	5.9e-3
18.2	θ_1	-1.100	-1.113	1.5e-1
	θ_2	0.280	0.293	1.4e-1
	θ_3	0.100	0.101	1.6e-2
	θ_4	0.080	0.078	1.7e-2

Conclusions and references

- The proposed three-stage parameter bounding procedure provides:
  - tight bounds on the parameters of the nonlinear block using steady-state input-output data;
  - overbounds on the parameters of the linear part, through the computation of bounds on the unmeasurable inner signal x_t;
- The numerical example has showed the effectiveness of the proposed procedure.
- The approach is computationally tractable (in the above example the experiment with $M = 50$ and $N = 1000$ required about 3 minutes (AMD 3200 processor)).

Reference paper:

V. Cerone, D. Regruto, “Parameter bounds evaluation of Wiener models with noninvertible polynomial non-linearities,” Automatica, vol. 42, pp. 1775–1781, 2002.